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Minimum Weight Design of Laminated Composite Plates Subject to Strength Constraint

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The minimum weight design of laminated composite plates subject to strength and side constraints is studied via a constrained global optimization technique. The first-ply failure load that is treated as the strength of a laminated plate is determined by using a shear deformable finite element and one of the several commonly used phenomenological failure criteria. The optimal layer group parameters (fiber angles and thicknesses of layer groups) of the laminated composite plate are determined via the proposed constrained global optimization technique for attaining the global minimum weight of the plate and satisfying the imposed constraints. A number of examples of the minimum weight design of symmetrically laminated composite plates with various aspect ratios, different number of layer groups, and different boundary conditions are given to illustrate the applications of the present constrained global optimal design method. The effects of the failure criteria on the optimal design parameters are also investigated via the examples. Finally, experimental investigation of the capability of the present method in obtaining global optima is performed. Failure tests of a number of graphite/epoxy laminates designed by different methods are performed, and the superiority of the present method over the other methods is demonstrated via the test results.

Introduction

LAMINATED composite plates or panels have found extensive applications in the construction of vehicle, aircraft, and spacecraft structures. In general, these structures are weight sensitive and reliability stringent. The design of lightweight but reliable laminated composite plates has thus become an important topic of research. The failure of composite materials is a complex process. Traditionally, the first-ply failure analysis based on some phenomenological failure criteria has been adopted in the safety design of laminated composite structures. The optimal design of laminated composite plates with minimum weight based on the traditional failure analysis technique has been studied by many researchers.[1–5] The previous work, however, is limited to the determination of a local minimum for laminated composite plates consisting of very few layers and thus has difficulties to find broad practical applications. Recently, the first author and his associates have developed an unconstrained multistart global optimal design method for enhancing the mechanical performance of laminated composite plates.[6–9] The proposed unconstrained optimization algorithm has been proved to be efficient and effective in designing laminated composite plates. The unconstrained optimization method has been successfully extended to the design of laminated composite plates with optimal dynamic characteristics subject to side constraints.[10] In this paper, the previously proposed unconstrained global optimization method is further extended to the minimum weight design of laminated composite plates subject to side and strength constraints. A number of phenomenological failure criteria are used in the minimum weight design and their effects on the optimal parameters of the laminated composite plates are studied. The feasibility and application of the present optimal design method are demonstrated by means of a number of examples of the minimum weight design of symmetrically laminated composite plates with different aspect ratios, numbers of layer groups, and boundary conditions. The factors that have important effects on the optimal design parameters and thicknesses of the plates are identified. The capability of the proposed optimal design method in obtaining global optima is validated by experimental results.

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Stress Analysis of Laminated Composite Plate

Consider a rectangular plate of area $a \times b$ and constant thickness h subject to transverse load $p(x, y)$, as shown in Fig. 1. The plate is composed of a finite number of layer groups in which each layer group contains several orthotropic layers of same fiber angle and uniform thickness. The x and y coordinates of the plate are taken in the midplane of the plate. The displacement field is assumed to be of the form

$$\begin{aligned} u_1(x, y, z) &= u_0(x, y) + z \cdot \psi_x(x, y) \\ u_2(x, y, z) &= v_0(x, y) + z \cdot \psi_y(x, y) \\ u_3(x, y, z) &= w(x, y) \end{aligned} \tag{1}$$

where u_1, u_2, and u_3 are displacements in the x, y, and z directions, respectively, and u_0, v_0, and w are the associated midplane displacements; ψ_x and ψ_y are shear rotations.

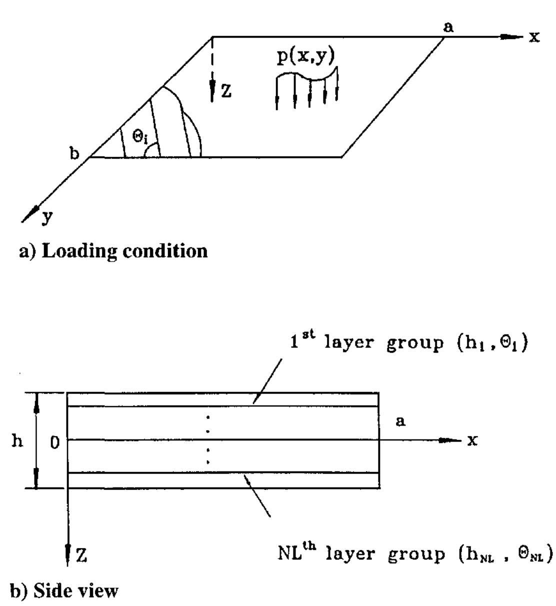


Fig. 1 Laminated composite plate.

The constitutive equations of a shear deformable laminated composite plate can be written as

$$\begin{bmatrix} N_1 \\ N_2 \\ Q_y \\ Q_x \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & 0 & A_{26} & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 & 0 \\ A_{16} & A_{26} & 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & 0 & 0 & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & 0 & 0 & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & 0 & 0 & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix}$$

$$\times \begin{bmatrix} u_{0,x} \\ v_{0,y} \\ w_{,y} + \psi_y \\ w_{,x} + \psi_x \\ u_{0,y} + v_{0,x} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \end{bmatrix} \tag{2}$$

where $N_1, N_2, \ldots, M_6$ are stress resultants; A_{ij}, B_{ij}, and D_{ij} are material components; and the comma before a subscript denotes the partial derivative with respect to the subscript. The material components are given by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}^{(m)}(1, z, z^2)\,dz \qquad (i, j = 1, 2, 6) \tag{3a}$$

and

$$A_{ij} = k_\alpha \cdot k_\gamma \cdot \bar{A}_{ij}, \qquad \bar{A}_{ij} = \int_{-h/2}^{h/2} Q_{ij}^{(m)}\,dz$$

$$i, j = 4, 5; \quad \alpha = 6 - i, \gamma = 6 - j \tag{3b}$$

where h is the total thickness of the plate. The stiffness coefficients $Q_{ij}^{(m)}$ depend on the material properties and orientation of the mth layer group. The parameters k_i are shear correction factors, which are determined using the expressions given by Whitney.[11]

In the stress analysis, the finite element developed by Kam and Chang[12] is adopted to evaluate stress distribution in the plate. The element contains five degrees of freedom (three displacements and two slopes, i.e., shear rotations) per node. In the evaluation of the element stiffness matrix, a quadratic element of a serendipity family and the reduced integration are used. The element has been used in the static analysis of both thin and moderately thick laminated composite plates and fairly good results for midplane displacements can be obtained.[12] The accuracy of the finite element in predicting strains was also investigated via an experimental approach.[13] A square $[0/90/0/90]_s$ graphite/epoxy laminate with clamped edges was loaded at the center, and strains at the center of the bottom surface of the laminate were measured in the experiment. The strain data were compared with those predicted by the finite element method. It was found that the use of a 3×3 mesh over a quarter-plate could yield very good results.

Strength of Composite Laminate

The strength P_c of a laminated composite plate is determined in the first-ply failure analysis of the plate using one of the phenomenological failure criteria, which are degenerate cases of the tensor polynomial criterion proposed by Tsai and Wu.[14] The introduction of the strength ratio β in the tensor polynomial criterion yields[15]

$$[F_{ij}\sigma_i\sigma_j]\beta^2 + [F_i\sigma_i]\beta - 1 = 0 \tag{4}$$

where F_{ij} and F_i are strength parameters. Note that a composite lamina under the preceding stress state is safe if $\beta > 1$. When $\beta \le 1$, the lamina is said to reach the limit state, and failure of the lamina occurs. The limit state equations of four phenomenological failure criteria for $\beta = 1$ are expressed as follows.[16]

1) Polynomial type maximum stress criterion: The polynomial type maximum stress criterion is expressed as

$$(\sigma_1 - X_T)(\sigma_1 + X_C)(\sigma_2 - Y_T)(\sigma_2 + Y_C)(\sigma_3 - Z_T)(\sigma_3 + Z_C)$$
$$\times (\sigma_4 - R)(\sigma_4 + R)(\sigma_5 - S)(\sigma_5 + S)(\sigma_6 - S)(\sigma_6 + S) = 0 \tag{5}$$

where σ_1, σ_2, and σ_3 are normal stress components; σ_4, σ_5, and σ_6 are shear stress components; X_T, Y_T, and Z_T are the lamina normal strengths in the 1, 2, and 3 directions; and R and S are the shear strengths in the 23 and 12 planes, respectively.

2) Hoffman's criterion: Hoffman's criterion is expressed as

$$\frac{1}{2}\left(-\frac{1}{X_TX_C} + \frac{1}{Y_TY_C} + \frac{1}{Z_TZ_C}\right)(\sigma_2 - \sigma_3)^2$$
$$+ \frac{1}{2}\left(\frac{1}{X_TX_C} - \frac{1}{Y_TY_C} + \frac{1}{Z_TZ_C}\right)(\sigma_3 - \sigma_1)^2$$
$$+ \frac{1}{2}\left(\frac{1}{X_TX_C} + \frac{1}{Y_TY_C} - \frac{1}{Z_TZ_C}\right)(\sigma_1 - \sigma_2)^2$$
$$+ \left(\frac{1}{X_T} - \frac{1}{X_C}\right)\sigma_1 + \left(\frac{1}{Y_T} - \frac{1}{Y_C}\right)\sigma_2 + \left(\frac{1}{Z_T} - \frac{1}{Z_C}\right)\sigma_3$$
$$+ \left(\frac{\sigma_4}{R}\right)^2 + \left(\frac{\sigma_5}{S}\right)^2 + \left(\frac{\sigma_6}{S}\right)^2 \ge 1 \tag{6}$$

3) Tsai–Hill criterion: The Tsai–Hill criterion is expressed as

$$\left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 + \left(\frac{\sigma_3}{Z}\right)^2 - \left(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2}\right)\sigma_1\sigma_2$$
$$- \left(-\frac{1}{X^2} + \frac{1}{Y^2} + \frac{1}{Z^2}\right)\sigma_2\sigma_3 - \left(\frac{1}{X^2} - \frac{1}{Y^2} + \frac{1}{Z^2}\right)\sigma_1\sigma_3$$
$$+ \left(\frac{\sigma_4}{R}\right)^2 + \left(\frac{\sigma_5}{S}\right)^2 + \left(\frac{\sigma_6}{S}\right)^2 \ge 1 \tag{7}$$

The values of X, Y, and Z are taken as either X_T, Y_T, and Z_T or as X_C, Y_C, and Z_C, depending upon the sign of σ_1, σ_2, and σ_3, respectively.

4) Tsai–Wu criterion: Based on the generalization of the von Mises criterion, the Tsai–Wu criterion is expressed as

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j \ge 1 \tag{8}$$

where

$$F_1 = (1/X_T) - (1/X_C); \qquad F_2 = (1/Y_T) - (1/Y_C)$$
$$F_3 = (1/Z_T) - (1/Z_C)$$
$$F_{11} = 1/X_TX_C; \qquad F_{22} = 1/Y_TY_C; \qquad F_{33} = 1/Z_TZ_C$$
$$F_{44} = 1/R^2; \qquad F_{55} = 1/S^2; \qquad F_{66} = 1/S^2 \tag{9}$$
$$F_{12} = -\frac{1}{2\sqrt{X_TX_CY_TY_C}}; \qquad F_{13} = -\frac{1}{2\sqrt{X_TX_CZ_TZ_C}}$$
$$F_{23} = -\frac{1}{2\sqrt{Y_TY_CZ_TZ_C}}$$

In the preceding failure criteria, the contribution of normal stress σ_3 is, in general, small compared with the other stress components so that its effects have been neglected. The shear strengths in the 12 and 13 planes are assumed to be the same. The strength ratios at the integration points as well as the node points for each layer group are computed, and the smallest value of the strength ratio among all of the layer groups is used for determining the strength of the plate.

Minimum Weight Design

Consider the minimum weight design of a laminated composite plate composed of NL layer groups subjected to the applied load $Pf(x, y)$, where P is the amplitude and $f(x, y)$ is shape function. The objective is to select the optimal fiber angles and thicknesses of the layer groups that yield the minimum weight of the plate. In mathematical form, the minimum weight design problem is stated as follows:

Minimize

$$W(\boldsymbol{h}, \boldsymbol{\theta}) = \sum_{i=1}^{NL} \rho A h_i$$

subject to

$$\begin{aligned} &0 \le \theta_i \le \pi, \qquad \theta_i = m_i\theta_0, \qquad h_i \ge 0 \\ &h_i = n_i h_0, \qquad \alpha P_c \ge \gamma P, \qquad i = 1, \ldots, NL \end{aligned} \tag{10}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{NL})^t$ and $\boldsymbol{h} = (h_1, h_2, \ldots, h_{NL})^t$ are vectors of fiber angles and thicknesses of layer groups, respectively; θ_0 and h_0 are production constants; W is plate weight; ρ is density; α and γ are resistance and load factors, respectively; and m_i and n_i are positive integers to be determined. The constraints on manufacturing composite laminates are considered via the equality constraints ($\theta_i = m_i\theta_0$, $h_i = n_i h_0$), and the weight of the plate is implicitly dependent on the fiber angles. If the strength of the plate is expressed in terms of the critical strength ratio β_c, which is the smallest among the strength ratios of the layer groups, the strength constraint $\alpha P_c \ge \gamma P$ in the preceding equation can be replaced by

$$(1/S_f)\beta_c \ge 1 \tag{11}$$

where the factor of safety

$$S_f = \gamma/\alpha \tag{12}$$

The solution of the preceding constrained minimization problem will be accomplished using the approach of two-level optimization. In the first-level optimization, the layer group parameters are treated as continuous design variables, and a constrained multistart global minimization method is presented for solving the preceding minimization problem and attaining the global minimum. In the second-level optimization, the final plate weight is determined by solving a discrete optimization problem in which the strength constraint as well as the manufacture constraints on layer group parameters are observed and the possible discrete values of the layer group parameters are obtained from the solution of the first-level optimization problem. The detailed description of the two-level optimization problem is given as follows.

First-Level Optimization

The preceding problem of Eq. (10) is first converted into an unconstrained minimization problem by creating the following general augmented Lagrangian[17]:

$$\Psi(\boldsymbol{\theta}, \boldsymbol{h}, \boldsymbol{\mu}, \eta, r_p) = W(\boldsymbol{\theta}, \boldsymbol{h}) + \sum_{j=1}^{NL} \left[\mu_j x_j + r_p x_j^2\right] + \left[\eta\phi + r_p\phi^2\right] \tag{13}$$

with

$$\begin{aligned} &x_j = \max[g_j(h_j), -\mu_j/2r_p] \\ &g_j(h_j) = -h_j \le 0 \qquad j = 1, \ldots, NL \\ &\phi = \max[H(\boldsymbol{\theta}, \boldsymbol{h}), -\eta/2r_p] \qquad H(\boldsymbol{h}, \boldsymbol{\theta}) = 1 - (1/S_f)\beta_c \le 0 \end{aligned} \tag{14}$$

where μ_j, η, and r_p are multipliers. Herein, the layer group parameters $\boldsymbol{\theta}$ and $\boldsymbol{h}$ are treated as continuous variables, and the equality constraints in Eq. (10) are disregarded in this level of optimization.

The update formulas for the multipliers μ_j, η, and r_p are

$$\begin{aligned} &\mu_j^{n+1} = \mu_j^n + 2r_p^n x_j^n \qquad j = 1, \ldots, NL \\ &\eta^{n+1} = \eta^n + 2r_p^n\phi^n \\ &r_p^{n+1} = \begin{cases} \gamma_0 r_p^n & \text{if} \quad r_p^{n+1} < r_p^{\max} \\ r_p^{\max} & \text{if} \quad r_p^{n+1} \ge r_p^{\max} \end{cases} \end{aligned} \tag{15}$$

where the superscript n denotes iteration number, γ_0 is a constant, and $r_p^{\max}$ is the maximum value of r_p. The parameters μ_j^0, η^0, r_p^0, γ_0, and $r_p^{\max}$ must be determined by the method of trial and error. From experience, the initial values of the multipliers and the values of the parameters (γ_0, $r_p^{\max}$) are chosen as

$$\begin{aligned} &\mu_j^0 = 1.0 \qquad j = 1, \ldots, NL \\ &\eta_0 = 1.0 \qquad r_p^0 = 0.4 \\ &\gamma_0 = 1.25 \qquad r_p^{\max} = 100 \end{aligned} \tag{16}$$

The minimum weight design problem of Eq. (13) has thus become the solution of the following unconstrained optimization problem:

Minimize

$$\Psi(\boldsymbol{\theta}, \boldsymbol{h}, \boldsymbol{\mu}, \eta, r_p) \qquad \text{with respect to } \boldsymbol{\theta} \text{ and } \boldsymbol{h}$$

subject to

$$0 \le \theta_i \le \pi \qquad i = 1, \ldots, NL \tag{17}$$

The preceding unconstrained optimization problem can be solved straightforwardly by using the previously proposed unconstrained multistart global optimization algorithm.[6,18] The basic idea of the unconstrained multistart global optimization method is to solve the problem of unconstrained minimization of a differentiable objective function $F(\boldsymbol{y}), \boldsymbol{y} \in \boldsymbol{Y} \subset R^n$ and $F \subset C^1$, with several local minima $\hat{\boldsymbol{F}}_j$ and corresponding local minimizers $\hat{\boldsymbol{y}}_j$. Note that, in the minimum weight design of laminated composite plates, as expressed in Eq. (17), $\boldsymbol{y}$ and $F(\boldsymbol{y})$ become $[\boldsymbol{\theta}, \boldsymbol{h}]$ and $\Psi(\boldsymbol{\theta}, \boldsymbol{h})$, respectively. In the global minimization process, a series of starting points are selected at random from the region of interest and a local minimization algorithm is used from each starting point. The search trajectories used by the local minimization algorithm are derived from the equation of motion of a particle of unit mass in an n-dimensional conservative force field, where the potential energy of the particle is represented by $F[\boldsymbol{y}(t)]$. In such a field the total energy of the particle, consisting of its potential and kinetic energies, is conserved. The motion of the particle is simulated, and by monitoring its kinetic energy an interfering strategy is adopted that ensures that potential energy is systematically reduced. In this way the particle is forced to follow a trajectory towards a local minimum in potential energy, $\hat{\boldsymbol{y}}$. Note that if the trajectory leaves the domain of interest at the point $[\boldsymbol{\theta}_p, h]$ where one or more of the components θ_{pi} take on values such that either $\theta_{pi} > \pi$ or $\theta_{pi} < 0$, then the constraints are imposed by continuing the trajectory at the point $[\boldsymbol{\theta}'_p, h]$ with components identical to $[\boldsymbol{\theta}_p, h]$ except for the components corresponding to the violated constraints. These components are replaced as follows:

$$\begin{aligned} &\theta'_{pi} = \theta_{pi} - \hat{m}\pi \qquad \text{if} \quad \theta_{pi} > \pi \\ &\theta'_{pi} = \theta_{pi} + \hat{m}\pi \qquad \text{if} \quad \theta_{pi} < 0; \qquad \hat{m} = 1, 2, 3, \ldots \end{aligned} \tag{18}$$

Here the value of $\hat{m}$ is chosen in such a way that θ'_{pi} satisfies the constraints. By not interrupting the motion of the particle with conserved total energy, other lower local minima, including, in particular, the global minimum, are obtained and recorded when the particle is traveling along its path. The motion of the particle is stopped once a termination criterion is satisfied. The same procedure is applied to the other starting points. As the process of searching for the global minimum continues, a Bayesian argument is used to establish the probability of the current overall minimum value of F being the global minimum, given the number of starts and number of times this value has been achieved. The multistart procedure is terminated once a target probability, typically 0.998, has been exceeded. In general, only a few starting points (less than 10) and a small number of iterations (fewer than 10) for each starting point are required to locate the global optimum. Note that the gradient of the critical strength ratios β_c required in the optimization algorithm can be obtained by taking differentiation of Eq. (4) with respect to the design variables:

$$\frac{d\beta_c}{dy_k} = \left\{-\beta_c^2 \frac{d}{dy_k}[F_{ij}\sigma_i\sigma_j] - \beta_c \frac{d}{dy_k}[F_i\sigma_i]\right\} \Big/ \{2\beta_c[F_{ij}\sigma_i\sigma_j] + F_i\sigma_i\} \tag{19}$$

where

$$y_k = \theta_k \quad \text{or} \quad h_k$$

The derivatives of stresses with respect to the design variable are well known and can be found in the literature.[19,20]

Second-Level Optimization

Once the global optimal solution of the first-level optimization has been obtained, the total number of plies N in the laminate can be determined as

$$N = \begin{cases} \text{Int}[h/h_0] & \text{for} \quad h/h_0 = \text{Int}[h/h_0] \\ \text{Int}[h/h_0] + 1 & \text{for} \quad h/h_0 > \text{Int}[h/h_0] \end{cases} \tag{20}$$

where Int[·] denotes the integer part of the number in the bracket. The possible values of multipliers m_i and n_i for fiber angle and thickness, respectively, of each layer group are determined as

$$m_i = \text{Int}[\theta_i/\theta_0] \quad \text{or} \quad \text{Int}[\theta_i/\theta_0] + 1 \tag{21a}$$

and

$$n_i = \text{Int}[h_i/h_0] \quad \text{or} \quad \text{Int}[h_i/h_0] + 1; \qquad i = 1, NL \tag{21b}$$

The second-level optimization problem then becomes the evaluation of the discrete layer group parameters $(\tilde{\theta}_i, \tilde{h}_i)$ that will minimize

$$W(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{h}})$$

subject to

$$P_c \geq S_f P \qquad N = \sum_{i=1}^{NL} n_i \tag{22}$$

where $\tilde{\theta}_i = m_i\theta_0$ and $\tilde{h}_i = n_i h_0$.

The solution of the preceding optimization problem can be easily achieved via the branch and bound approach.[21,22] In the present study, for small θ_0, say $\theta_0 \leq 10$ deg, the design space is very flat with respect to fiber angles in the vicinity of the optimum, and small perturbations in θ_i do not affect the final optimum plate weight. The final fiber angles can then be determined immediately using the method of rounding in which the fiber angles obtained in the first-level optimization are rounded off to the nearest available discrete values given in Eq. (21a). If the number of layer groups is small, the final layer group thicknesses can be easily determined via the process of enumeration in which every possible combination of the multipliers n_i determined from Eq. (21b) is tested and the best optimum is selected subject to the constraints in Eq. (22).

Example

Before proceeding to the minimum weight design of laminated composite plates, it is worthwhile demonstrating the accuracy of the present finite element method in determining the first-ply failure loads of laminated composite plates. The first-ply failure loads of various clamped laminated composite plates made of T300/5208 graphite/epoxy (see Table 1 for laminate properties) subjected to a uniform load have been determined via the present method using a 2 × 2 mesh over a quarter-plate and verified by those reported in the literature. For instance, the first-ply failure loads obtained by the present method on the basis of different failure criteria for a [0/90]s plate are listed in Table 2 in comparison with those obtained by Reddy and Reddy[16] in which a 6 × 3 mesh over a quarter-plate was used. The first-ply failure loads obtained by the present finite element method closely match those obtained by the previous researchers even when fewer number of elements are used in the analysis. The failure locations predicted by the present method are also identical to those predicted by Reddy and Reddy.[16] The capability of the present optimal design method in obtaining the global optimal design for composite laminates subjected to strength constraint has also been verified by the results reported in the literature. For instance, the present method can yield the same global optima for laminates subjected to a point force as obtained by Tauchert and Adibhatla[23] using a different optimization technique. Therefore, the present method will be used to perform the minimum weight design of symmetrically laminated composite plates with different aspect ratios ($b/a = 0.5, 1.0, 1.2$), numbers of layer groups ($NL = 4, 6, 8$) and boundary conditions (simple supports or clamped edges), on the basis of four different failure criteria and safety factor $S_f = 1$. The plates are made of T300/5208 graphite/epoxy composite material (see Table 1 for material properties and strengths) and subjected to a uniform load of intensity $P = 108$ N/cm^2. The boundary conditions of simple supports and clamped edges are shown in Fig. 2. The advantages of using the present optimal design method are first demonstrated by means of the design of simply supported symmetrically laminated composite plates composed of four layer groups with various aspect ratios ($a = 10$ cm, $b/a = 0.5, 1.0, 1.2$). For illustration purpose, the production constants of fiber angle and layer thickness are set as $\theta_0 = \pi/36$ rad(5 deg) and $h_0 = 0.015$ cm, respectively. For comparison, Table 3 lists the plate thicknesses designed on the basis of the Tsai–Hill failure criterion using three different methods, namely, 1) fiber angles are set as $[0/90]_s$ and layer groups are of equal thickness (method 1), 2) fiber angles are

Table 2 First-ply failure of a $[0/90]_s$ plate with clamped edges

Failure criterion	Normalized failure load $(P/E_2)(a/h)^4$ Present (1)	Reddy and Reddy[16] (2)	Difference (2) − (1)/(2)%
Maximum stress (i)[a]	18,827.7	19,031.8	1.07
Maximum strain (i)	18,900.8	19,105.5	1.07
Maximum stress (p)[b]	18,821.1	19,025.1	1.07
Maximum strain (p)	18,901.5	19,103.7	1.06
Hoffman	18,828.4	19,025.8	1.04
Tsai–Hill	18,828.4	19,032.2	1.07
Tsai–Wu	18,872.3	19,050.9	0.94

[a]The term (i) stands for independent. [b]The term (p) stands for polynomial.

Table 1 Properties of T300/5208 graphite/epoxy composite laminates

Properties	Values	Properties	Values
E_1	19.2 × 10^6 psi (132.5 Gpa)	X_T	219.5 × 10^3 psi (1515 Mpa)
E_2	1.56 × 10^6 psi (10.8 Gpa)	X_C	246.0 × 10^{3}psi (1697 Mpa)
E_3	1.56 × 10^6 psi (10.8 Gpa)	$Y_T = Z_T$	6.35 × 10^3 psi (43.8 Mpa)
$G_{12} = G_{23}$	0.82 × 10^6 psi (5.7 Gpa)	$Y_C = Z_C$	6.35 × 10^3 psi (43.8 Mpa)
G_{23}	0.49 × 10^6 psi (3.4 Gpa)	R	9.80 × 10^3 psi (67.6 Mpa)
$\nu_{12} = \nu_{13}$	0.24	$S = T$	12.6 × 10^3 psi(86.9 Mpa)
ν_{23}	0.49	Ply thickness h_i, in.	0.005
		a, in.	9
		b, in.	5

Table 3 Simply supported symmetrically laminated plates designed by different methods (h_0 = 0.015 cm, a = 10 cm, and P = 108 N/cm^3)

b/a	Design method	Layup, deg	Layer group ply numbers $[n_1/n_2]_s$	First-ply failure load P_c, N/cm^2	Specific strength (P_c/h)
	1	$[0/90]_s$	$[7/7]_s$	119.6	284.9
0.5	2	$[75/-55]_s$	$[3/3]_s$	108.0	600.0
	Present	$[75/-65]_s$	$[1/5]_s$	115.4	641.1
	1	$[0/90]_s$	$[7/7]_s$	111.0	264.3
1.0	2	$[30/-45]_s$	$[6/7]_s$	117.5	301.2
	Present	$[45/-45]_s$	$[2/9]_s$	117.0	346.9
	1	$[0/90]_s$	$[6/7]_s$	114.5	293.5
1.2	2	$[0/-75]_s$	$[6/7]_s$	115.7	296.7
	Present	$[30/-35]_s$	$[2/10]_s$	124.9	346.9

Table 4 Optimal solutions for simply supported and uniformly loaded laminated composite plates designed for minimum weight (h_0 = 0.015 cm, a = 10 cm, and P = 108 N/cm^2)

		Number of layer groups, NL								
		4			6			8		
Aspect ratio b/a	Failure criterion[a]	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2]_s$	Specific strength (P_c/h)	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2/n_3]_s$	Specific strength (P_c/h)	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2/n_3/n_4]_s$	Specific strength (P_c/h)
	I	$[70/-65]_s$	$[1/5]_s$	631.1	$[70/-65/85]_s$	$[1/4/1]_s$	674.6	$[-90/65/55/-55]_s$	$[1/1/1/3]_s$	675.6
0.5	II	$[85/-65]_s$	$[1/5]_s$	634.4	$[85/-75/65]_s$	$[1/1/4]_s$	678.0	$[85/-75/65/90]_s$	$[1/1/4/0]_s$	678.0
	III	$[75/-65]_s$	$[1/5]_s$	642.2	$[90/-60/60]_s$	$[1/1/4]_s$	688.0	$[90/-60/60/90]_s$	$[1/1/4/0]_s$	688.0
	IV	$[75/-65]_s$	$[1/5]_s$	697.8	$[90/-60/55]_s$	$[1/1/4]_s$	771.6	$[90/-75/55/-55]_s$	$[1/1/1/3]_s$	783.8
	I	$[45/-45]_s$	$[2/9]]_s$	349.4	$[45/-45/45]_s$	$[1/4/6]_s$	351.0	$[45/-45/-45/45]_s$	$[1/2/2/6]_s$	351.0
1.0	II	$[45/-45]_s$	$[2/9]_s$	350.5	$[45/-45/45]_s$	$[1/4/6]_s$	352.1	$[45/-45/-45/45]_s$	$[1/2/2/6]_s$	352.1
	III	$[45/-45]_s$	$[2/9]_s$	354.4	$[45/-45/45]_s$	$[1/4/6]_s$	356.1	$[45/-45/-45/45]_s$	$[1/2/2/6]_s$	356.1
	IV	$[45/-45]_s$	$[2/9]_s$	364.2	$[-25/-50/45]_s$	$[1/1/9]_s$	367.9	$[25/-55/-45/45]_s$	$[1/1/2/7]_s$	379.1
	I	$[25/-35]_s$	$[2/10]_s$	339.7	$[25/-35/30]_s$	$[2/9/1]_s$	340.3	$[0/-20/35/-35]_s$	$[1/1/2/8]_s$	343.1
1.2	II	$[30/-35]_s$	$[2/10]_s$	342.5	$[-30/35/-45]_s$	$[2/9/1]_s$	343.5	$[-10/30/-35/35]_s$	$[1/2/8/1]_s$	348.1
	III	$[30/-35]_s$	$[2/10]_s$	346.9	$[-15/-35/35]_s$	$[1/2/9]_s$	348.9	$[0/-20/35/-35]_s$	$[1/1/2/8]_s$	356.1
	IV	$[30/-30]_s$	$[2/10]_s$	353.9	$[10/-45/45]_s$	$[2/2/8]_s$	373.3	$[10/-25/-50/40]_s$	$[2/1/1/8]_s$	381.7

[a] I = polynomial type maximum stress criterion, II = Hoffman criterion, III = Tsai–Hill criterion, and IV = Tsai–Wu criterion.

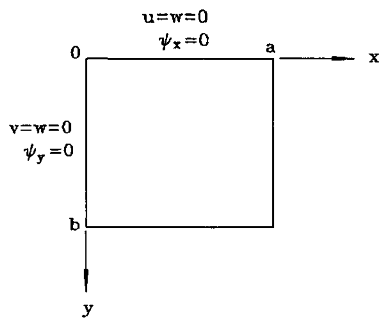
Table 5 Optimal solutions for clamped and uniformly loaded laminated composite plates designed for minimum weight (h_0 = 0.015 cm, a = 10 cm, and P = 108 N/cm^2)

		Number of layer groups, NL								
		4			6			8		
Aspect ratio b/a	Failure criterion[a]	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2]_s$	Specific strength (P_c/h)	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2/n_3]_s$	Specific strength (P_c/h)	Fiber angle θ_i, deg	Layer group ply number $[n_1/n_2/n_3/n_4]_s$	Specific strength (P_c/h)
	I	$[90/0]_s$	$[2/3]_s$	912.6	$[90/0/90]_s$	$[2/2/1]_s$	925.0	$[90/0/90/90]_s$	$[1/2/1/1]_s$	925.0
0.5	II	$[90/0]_s$	$[2/3]_s$	912.8	$[90/0/90]_s$	$[2/2/1]_s$	928.3	$[90/0/90/90]_s$	$[1/2/1/1]_s$	928.3
	III	$[90/0]_s$	$[2/3]_s$	912.8	$[90/0/90]_s$	$[2/2/1]_s$	937.2	$[90/90/0/90]_s$	$[1/1/2/1]_s$	937.2
	IV	$[90/0]_s$	$[2/3]_s$	914.4	$[90/0/90]_s$	$[2/2/1]_s$	940.0	$[90/90/0/90]_s$	$[1/1/2/1]_s$	940.0
	I	$[40/-60]_s$	$[2/7]_s$	408.2	$[45/-35/90]_s$	$[1/2/6/]_s$	430.4	$[-45/-50/55/0]_s$	$[1/1/1/6]_s$	430.7
1.0	II	$[50/-30]_s$	$[2/7]_s$	410.4	$[-45/35/90]_s$	$[1/2/6]_s$	423.4	$[45/-35/-20/90]_s$	$[1/1/1/6]_s$	439.8
	III	$[45/-35]_s$	$[2/7]_s$	412.9	$[45/-55/0]_s$	$[1/2/6]_s$	425.7	$[45/-35/-20/90]_s$	$[1/1/1/6]_s$	447.1
	IV	$[45/-45]_s$	$[2/7]_s$	450.0	$[45/-50/-5]_s$	$[1/2/6]_s$	457.1	$[45/-50/-70/0]_s$	$[1/1/1/6]_s$	471.5
	I	$[45/-15]_s$	$[2/7]_s$	400.0	$[45/-40/0]_s$	$[1/2/6]_s$	400.5	$[-45/40/30/0]_s$	$[1/1/1/6]_s$	401.0
1.2	II	$[45/-15]_s$	$[2/7]_s$	402.1	$[-45/40/0]_s$	$[1/2/6]_s$	405.0	$[-45/40/0/0]_s$	$[1/2/1/5]_s$	405.0
	III	$[45/-15]_s$	$[2/7]_s$	403.7	$[45/-30/0]_s$	$[1/1/7]_s$	412.2	$[45/40/-45/0]_s$	$[1/1/1/6]_s$	413.1
	IV	$[45/-25]_s$	$[2/7]_s$	421.1	$[45/-40/-5]_s$	$[1/2/6]_s$	442.6	$[45/-40/-60/-5]_s$	$[1/1/1/6]_s$	443.1

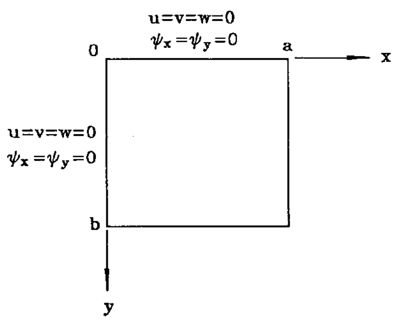
[a] I = polynomial type maximum stress criterion, II = Hoffman criterion, III = Tsai–Hill criterion, and IV = Tsai–Wu criterion.

treated as design variables whereas layer groups are of equal thickness (method 2), and 3) both fiber angles and layer group thicknesses are treated as design variables (present method). Note that a significant reduction in the number of plies and an increase in specific strength, i.e., the ratio of first-ply failure load to plate thickness, can be achieved when the plates are designed by using the present method. For instance, 16 plies have been saved, whereas on the other hand approximately a 125% increase in specific strength has been achieved for the plate of $b/a = 0.5$ when the design method changes from the first type (method 1) to the third type (present method). Next consider the minimum weight design of symmetrically laminated composite plates subjected to a uniform load of intensity $P = 108$ N/cm^2. The optimal solutions for plates with simple supports and clamped edges are listed in Tables 4 and 5, respectively. In Table 4, the optimal layer group parameters (fiber angles and numbers of plies) as well as the corresponding specific strengths determined on the basis of different failure criteria are tabulated for the plates with various aspect ratios (b/a = 0.5, 1.0, 1.2) and numbers of layer groups (NL = 4, 6, 8). Note that aspect ratio may have significant effects on the optimal design parameters of the plates, whereas the type of failure criterion chosen only has minimal or even no effects on them. For instance, for the case of $NL = 4$ in Table 4, the optimal fiber angles change from $[70/-65]_s$ to $[25/-30]_s$ and the optimal numbers of plies in the layer groups from $[1/5]_s$ to $[2/10]_s$ based on the polynomial type maximum stress criterion as aspect ratio varies from 0.5 to 1.2. Also note that, among the four criteria, the

Tsai–Wu criterion predicts the highest specific strength, whereas the polynomial type maximum stress criterion does the opposite. In general, the increase in layer group number (NL) may increase the specific strength of the laminated plates. The increase of specific strength, however, may converge or become insignificant when $NL \geq 6$. Therefore, the use of six layer groups for the design of symmetric laminates may yield reasonably good results. The use of a smaller number of layer groups in manufacturing composite laminates may greatly reduce the time for manufacture and thus the cost of production. The fact that the use of the present method in designing composite laminates can effectively and efficiently reduce the number of layer groups illustrates one of the important merits of the present method. Finally, consider the optimal solutions for the clamped symmetrically laminated composite plates. In Table 5, the optimal layer group parameters as well as the corresponding specific strengths determined on the basis of different failure criteria are tabulated for the clamped plates with various aspect ratios ($b/a = 0.5, 1.0, 1.2$) and numbers of layer groups ($NL = 4, 6, 8$). Similar to the simply supported plates, it has also been observed that aspect ratio may have significant effects on the optimal design parameters of the clamped plates, whereas the type of failure criterion only has slight or even no effects on them. For instance, for the case of $NL = 4$ in Table 5 the optimal fiber angles change from [90/0]s to [45/−25]s and the optimal layer group ply numbers from [2/3]s to [2/7]s based on the Tsai–Wu criterion as aspect ratio varies from 0.5 to 1.2. Again among the four failure criteria, the Tsai–Wu criterion predicts the highest specific strength, whereas the polynomial type maximum stress criterion does the opposite. The increase of specific strength may become insignificant or converge when $NL \geq 6$ and thus again the use of six layer groups may yield reasonably good results. Also note that the optimal design parameters of the simply supported plates are different from those of the clamped ones and the specific strength of the clamped plates may be much higher than that of the simply supported plates when they have the same dimensions and number of plies. For instance, for $b/a = 0.5$ and $NL = 4$ on the basis of the Tsai–Wu criterion the optimal fiber angles for the simply supported and the clamped plates are [75/−65]s in Table 4 and [90/0]s in Table 5, respectively; the optimal layer group ply numbers for the simply supported and the clamped plates are [1/5]s and [2/3]s, respectively; the specific strength of the clamped plate (914.4 N/cm^3) is about 30% higher than that of the simply supported one (697.8 N/cm^3).



a) Simple support



b) Fixed edges

Fig. 2 Boundary conditions of laminated composite plates.

Experimental Verification

The present method is first used to design clamped square (10×10 cm) 24-ply symmetric graphite/epoxy laminates under the action of a center point load. The properties of the graphite/epoxy composite material are obtained experimentally in accordance with the relevant American Society for Testing and Material (ASTM) standards and are given in Table 6. The objective of the optimal design is to determine the optimal design parameters (layer group thicknesses and fiber angles) for attaining the maximum first-ply failure loads of the laminates with same number of plies but composed of different numbers of layer groups. Using the idea proposed in the previous

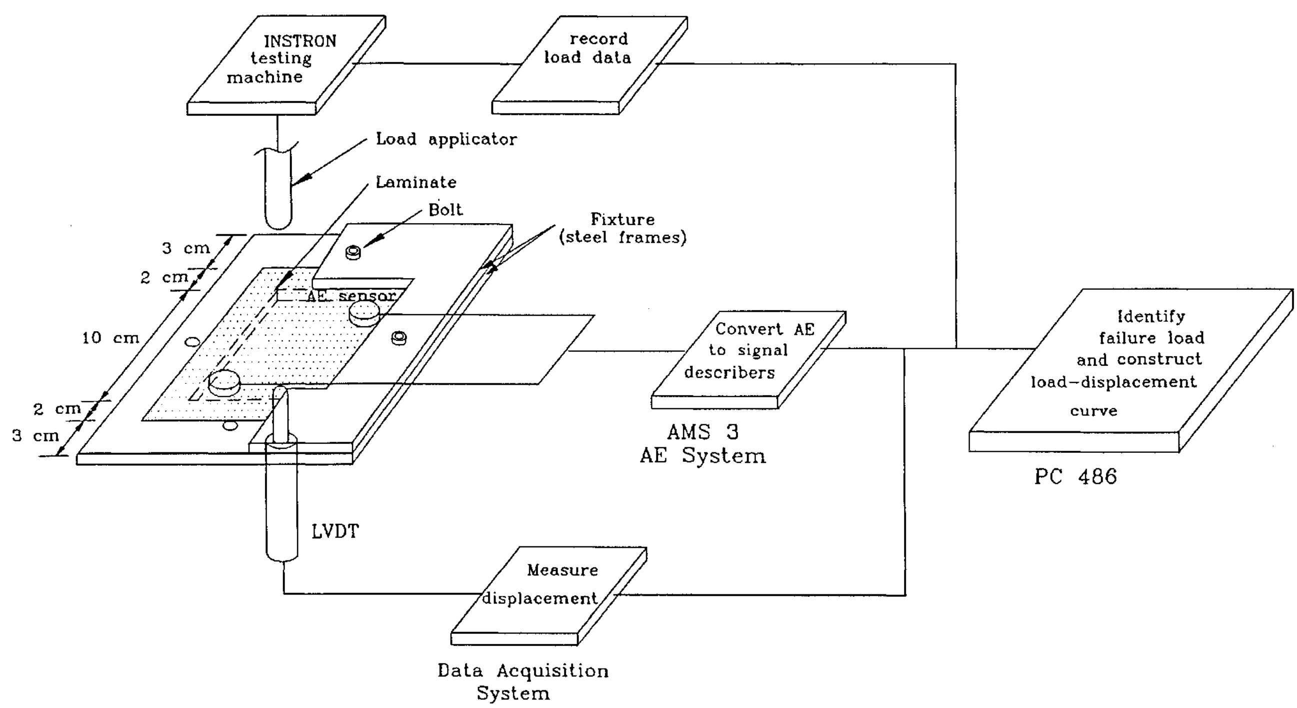


Fig. 3 Schematic description of the experimental setup.

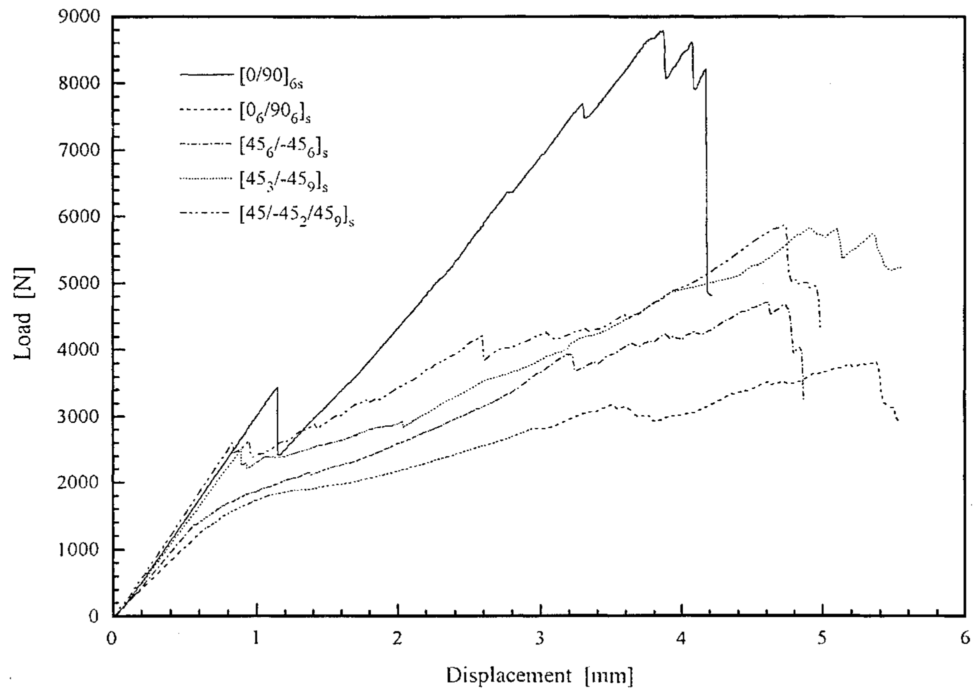


Fig. 4 Load-displacement curves of laminates designed by various methods.

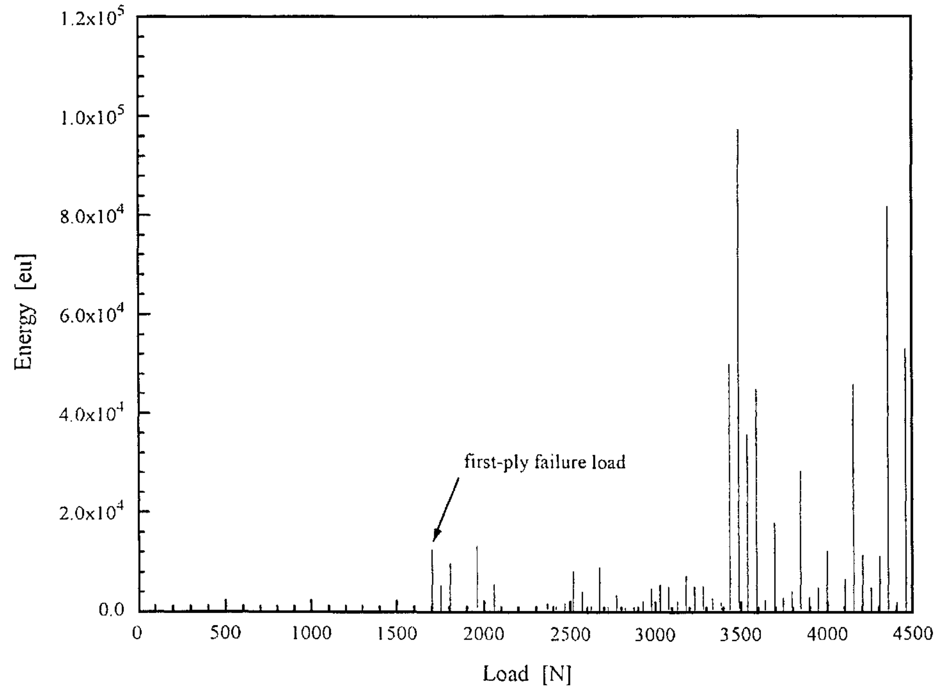


Fig. 5 Energy vs load produced by AMS3 acoustic emission system for a $[0/90]_{6s}$ laminate.

Table 6 Material properties of graphite/epoxy

Material constants		Strengths			
E_1, Gpa	132.5	X_T, Gpa	1690.75	$X_{\varepsilon T}$	0.012925
E_2, Gpa	7.90	X_C, Gpa	1893.64	$X_{\varepsilon C}$	0.014475
E_3, Gpa	7.90	$Y_T = Z_T$, Gpa	41.06	$Y_{\varepsilon T} = Z_{\varepsilon T}$	0.003729
$G_{12} = G_{13}$, Gpa	4.20	$Y_C = Z_C$, Gpa	41.06	$Y_{\varepsilon C} = Z_{\varepsilon C}$	0.003729
G_{23}, Gpa	1.02	R, Gpa	46.1	R_ε	0.038674
$\nu_{12} = \nu_{13}$	0.28	$S = T$, Gpa	58.7	$S_\varepsilon = T_\varepsilon$	0.014191
ν_{23}	0.25				
h_0, cm	0.015				

sections and based on the Tsai–Wu failure criterion, the optimal design parameters as well as the maximum first-ply failure loads are determined by choosing $\theta_0 = \pi/36$ rad (5 deg) and $h_0 = 0.015$ cm, and the results are listed in Table 7 for $NL = 4$, 6, and 8. The maximum first-ply failure load converges when $NL = 6$.

For experimental investigation, a number of square 24-ply symmetric laminates, namely, $[0/90]_{6s}$, $[0_6/90_6]_s$, $[45_6/-45_6]_s$, $[45_3/-45_9]_s$, and $[45/-45_2/45_9]_s$ of dimensions 14 × 14 cm designed by using different design methods were manufactured and tested to complete failure in accordance with the test procedure described in Ref. 13. A schematic description of the experimental setup

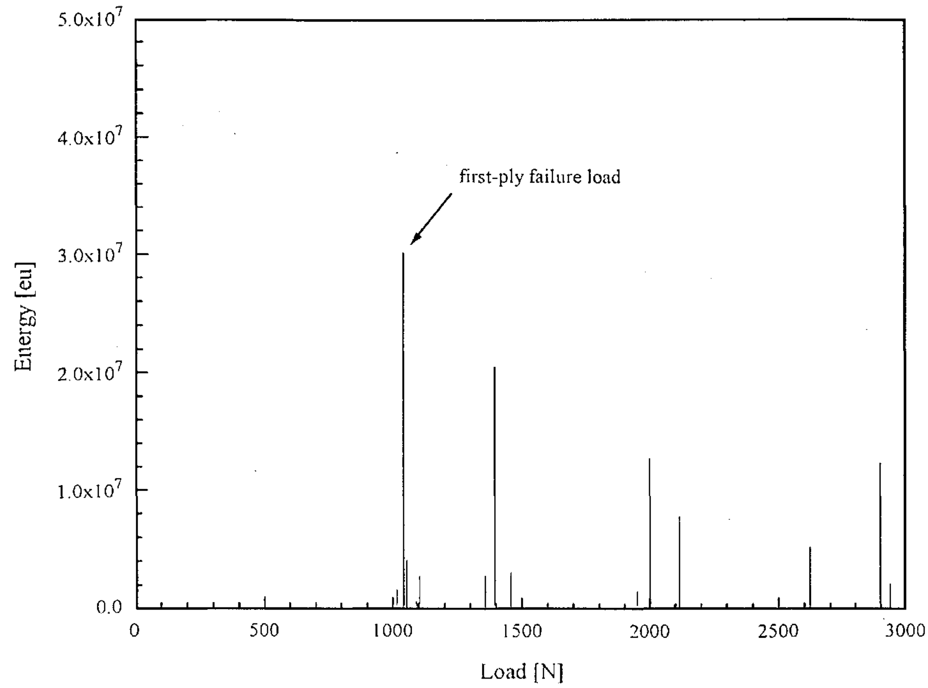


Fig. 6 Energy vs load produced by AMS3 acoustic emission system for a $[0_6/90_6]_s$ laminate.

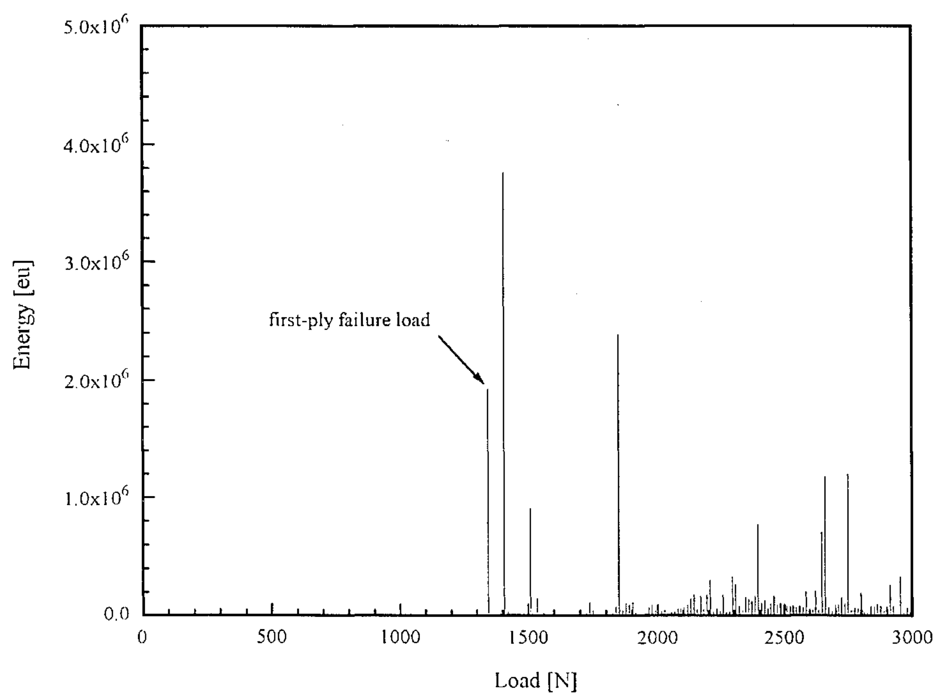


Fig. 7 Energy vs load produced by AMS3 acoustic emission system for a $[45_6/-45_6]_s$ laminate.

Table 7 Design of a square 24-ply symmetric laminate

Layer groups	Layup, deg	First-ply failure load, N
4	$[45_3/-45_9]_s$	1703.1
6	$[45/-45_2/45_9]_s$	1968.1
8	$[45/-45_2/45/45_8]_s$	1968.1

is shown in Fig. 3 in which the laminate is clamped at all edges and the actual dimensions of the laminate are 10 × 10 cm. The load-center displacement curves of the laminates shown in Fig. 4 are constructed using the data measured from the displacement transducer (LVDT) placed beneath the center of the bottom surface of the laminate. The first-ply failure loads of the laminate were determined from the energy-applied load diagrams produced by the AMS3 acoustic emission system. Figures 5–9 show the energy-applied load diagrams of the laminated composite plates from which the first-ply failure loads can be easily identified. The theoretical and experimental first-ply failure loads of the laminates are listed in Table 8 for comparison. Note that in addition to the aforementioned validation of the present method, the fact that the $[45/-45_2/45_9]_s$ yields the highest first-ply failure load further verifies the capability of the present method in producing global optima. Although the $[0/90]_{6s}$ laminate cannot yield the highest first-ply failure load, it can yield the highest collapse load amongst the laminates. In general, a laminate can substain much higher load after the first-ply failure. Therefore, how to incorporate the load-carrying capacity of a laminate beyond its first-ply failure in the safety design of the laminate should become an important topic for future research.

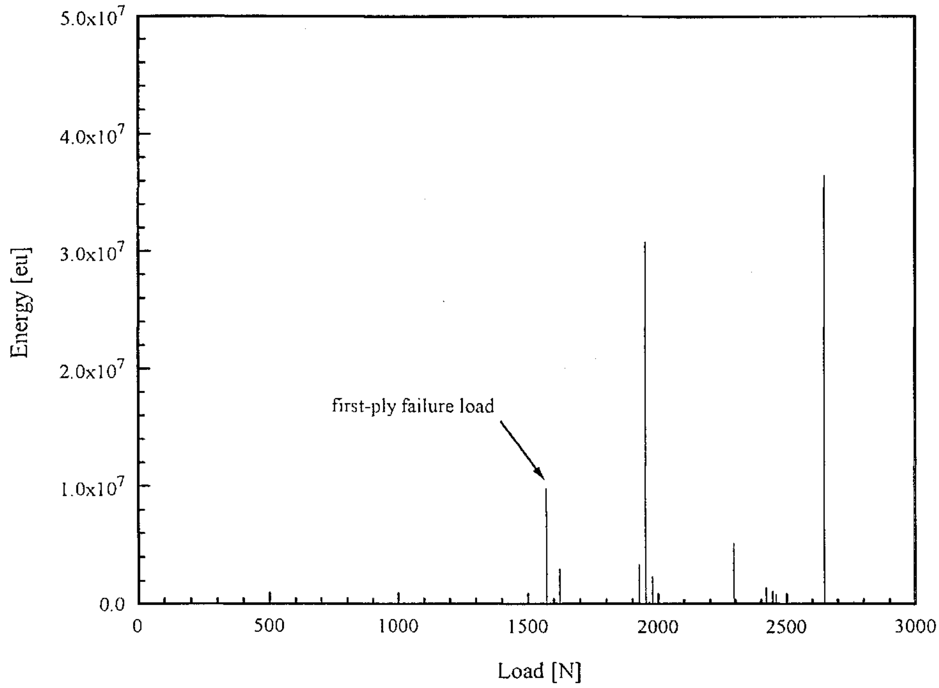


Fig. 8 Energy vs load produced by AMS3 acoustic emission system for a $[45_3/-45_9]_s$ laminate.

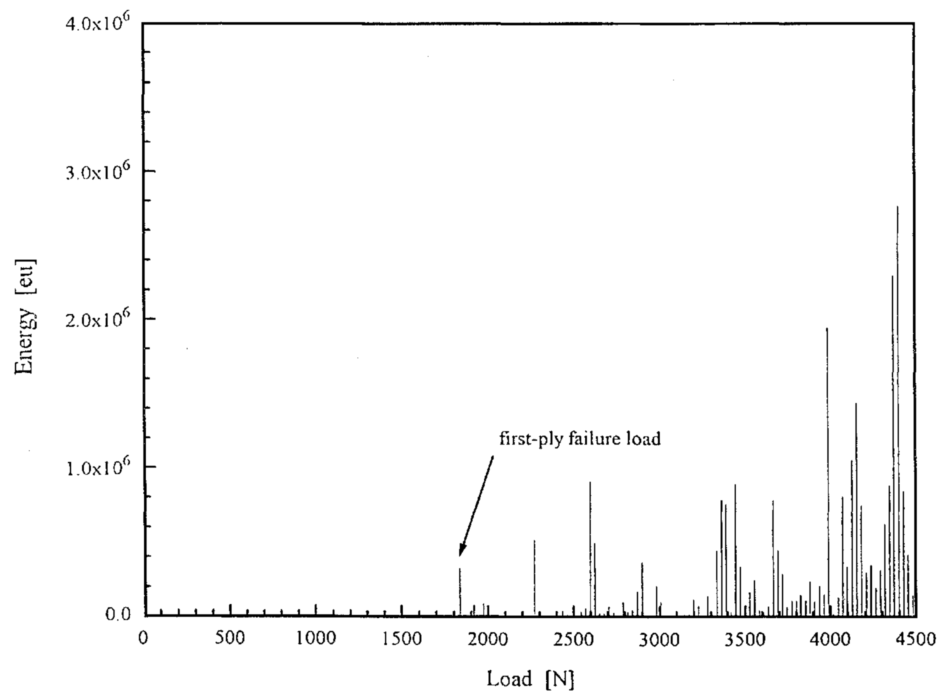


Fig. 9 Energy vs load produced by AMS3 acoustic emission system for a $[45/-45_2/45_9]_s$ laminate.

Table 8 First-ply failure loads of laminates designed by various methods

	First-ply failure load, N		
Laminate	Theoretical (1)	Experimental (2)[a]	Error |(2)−(1)/(2)|%
$[0/90]_{6s}$	1794.5	1702.9	5.38
$[0_6/90_6]_s$	1250.5	1041.5	20.1
$[45_6/-45_6]_s$	1279.2	1345.1	4.90
$[45_3/-45_9]_s$	1703.1	1572.7	8.32
$[45/-45_2/45_9]_s$[b]	1968.1	1835.7	7.21

[a]Average value of three specimens.
[b]Laminate possesses the highest first-ply failure load.

Conclusion

A constrained global optimization method was developed for the minimum weight design of laminated composite plates subject to side and strength constraints. Optimal design parameters of symmetrically laminated composite plates of various aspect ratios, numbers of layer groups, and boundary conditions subjected to uniform loads were determined on the basis of several phenomenological failure criteria. The effects of aspect ratio, number of layer groups, and boundary conditions on the optimal lamination arrangements as well as the effects of the type of failure criterion and number of layer groups on plate thickness were investigated. It has been found that aspect ratio and boundary conditions have significant effects on the optimal design parameters, whereas the type of failure criterion only has slight effects on them. Furthermore, both the type of failure criterion and number of layer groups have insignificant effects on plate thickness. In general, among the adopted failure criteria plates designed on the basis of the Tsai–Wu failure criterion possess the highest specific strength or lightest weight and the use of six layer groups in the design can yield plates with weight close to the global optimum value. Experiments of laminates designed by various methods and subjected to a center point load were performed. Comparison between theoretical and experimental results was carried out to validate the present method. Thus the results presented in this paper should be valuable for practical applications.

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References

[1]Khot, N. S., "Computer Program (OPTCOMP) for Optimization of Composite Structures for Minimum Weight Design," Air Force Flight Dynamics Lab., AFFDL TR-76-149, Wright–Patterson AFB, OH, June 1977.

[2]Schmit, L. A., and Farshi, B., "Optimum Design of Laminated Composite Plates," *International Journal of Numerical Methods in Engineering*, Vol. 11, No. 4, 1977, pp. 623–640.

[3]Park, W. J., "An Optimal Design of Simple Symmetric Laminates Under First Ply Failure Criterion," *Journal of Composite Materials*, Vol. 16, No. 4, 1982, pp. 341–345.

[4]Watkins, R. J., and Morris, A., "A Multicriteria Objective Function Optimization Scheme for Laminated Composites for Use in Multilevel Structural Optimization Schemes," *Journal of Computer Methods in Applied Mechanics and Engineering*, Vol. 60, No. 2, 1987, pp. 233–251.

[5]Kam, T. Y., "Optimum Design of Laminated Composite Structures via a Multilevel Substructuring and Optimization Approach," *Engineering Optimization*, Vol. 19, No. 2, 1992, pp. 81–100.

[6]Kam, T. Y., and Snyman, J., "Optimal Design of Laminated Composite Structures Using a Global Optimization Technique," *Journal of Composite Structure*, Vol. 19, No. 4, 1991, pp. 351–370.

[7]Kam, T. Y., and Chang, R. R., "Design of Laminated Composite Plates for Maximum Axial Buckling Load and Vibration Frequency," *Journal of Computer Methods in Applied Mechanics and Engineering*, Vol. 106, No. 1, 1993, pp. 65–81.

[8]Kam, T. Y., and Chang, R. R., "Optimal Layup of Thick Laminated Composite Plates for Maximum Stiffness," *Engineering Optimization*, Vol. 19, No. 3, 1992, pp. 237–249.

[9]Chang, R. R., Chu, G. H., and Kam. T. Y., "Design of Laminated Composite Plates for Maximum Shear Buckling Load," *Journal of Energy Resources Technology*, Vol. 115, No. 4, 1993, pp. 314–322.

[10]Kam, T. Y., and Lai, F. M., "Design of Laminated Composite Plates for Optimal Dynamic Characteristics Using a Constrained Global Optimization Technique," *Journal of Computer Methods Applied in Mechanics and Engineering*, Vol. 120, No. 3–4, 1995, pp. 389-402.

[11]Whitney, J. M., "Shear Correction Factors for Orthotropic Laminates Under Static Load," *Journal of Applied Mechanics*, Vol. 40, March 1973, pp. 302–304.

[12]Kam, T. Y., and Chang, R. R., "Finite Element Analysis of Shear Deformable Laminated Composite Plates," *Journal of Energy Resources and Technology*, Vol. 115, No. 1, 1993, pp. 41–46.

[13]Kam, T. Y., Sher, H. F., Chao, T. N., and Chang, R. R., "Predictions of Deflection and First-Ply Failure Load of Thin Laminated Composite Plates via the Finite Element Approach," *Journal of Solids Structures*, Vol. 33, No. 3, 1996, pp. 375–398.

[14]Tsai, S. W., and Wu, E. M., "A General Theory of Strength for Anisotropic Materials," *Journal of Composite Materials*, Vol. 5, No. 1, 1971, pp. 58–80.

[15]Tsai, W. S., *Introduction to Composite Materials*, Technomic, Westport, CT, 1980.

[16]Reddy, Y. S. N., and Reddy, J. N., "Linear and Non-Linear Failure Analysis of Composite Laminates with Transverse Shear," *Composites Science and Technology*, Vol. 44 , No. 4, 1992, pp. 227–255.

[17]Vanderplaats, G. N., *Numerical Optimization Techniques for Engineering Design: With Applications*, McGraw–Hill, New York, 1984.

[18]Snyman, J. A., and Fatti, L. P., "A Multi-Start Global Minimization Algorithm with Dynamic Search Trajectories," *Journal of Optimization Theory and Applications*, Vol. 54, No. 1, 1987, pp. 121–141.

[19]Kirsch, U., *Optimum Structural Design*, McGraw–Hill, New York, 1981.

[20]Venkayya, V. B., "Design of Optimum Structures," *Computer and Structures*, Vol. 1, No. 1/2, 1971, pp. 265–309.

[21]Dakin, R. J., "A Tree Search Algorithm for Mixed Integer Programming Problems," *Computer Journal*, Vol. 8, No. 3, 1965, pp. 250–255.

[22]Mesquita, L., and Kamat, M. P., "Optimization of Stiffened Laminated Composite Plates with Frequency Constraints," *Engineering Optimization*, Vol. 11, No. 1, 1987, pp. 77–88.

[23]Tauchert, T. R., and Adibhatla, S., "Design of Laminated Plates for Maximum Bending Strength," *Engineering Optimization*, Vol. 8, Aug. 1985, pp. 253–263.